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BMN operators from Wilson loop

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Abstract: We show that the BMN operators arise from the expansion of the Wilson loop in four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. The Wilson loop we consider is obtained from "dimensional reduction" of ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory, and it contains six scalar fields as well as the gauge field. We expand the Wilson loop twice. First we expand it in powers of the fluctuations around a BPS loop configuration. Then we further expand each term in the result of the first step in powers of the scalar field Z associated with the BPS configuration. We find that each operator in this expansion with large number of Z is the BMN operator. The number of fluctuations corresponds to the number of impurities, and the phase factor of each BMN operator is supplied correctly. We have to impose the locally supersymmetric condition on the loop for obtaining the complete form of the BMN operators including the correction terms with $\bar{Z}$. Our observation suggests the correspondence between the Wilson loop and the string field.

Keywords: AdS-CFT and dS-CFT Correspondence, Penrose limit and pp-wave background.

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Contents

1. Introduction

The AdS/CFT correspondence [1−5] is a conjecture that there is an equivalence between type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and four-dimensional $\mathcal{N} = 4$ SU(N) super Yang-Mills theory (SYM) on the boundary of AdS_5. This correspondence was first proposed by Maldacena [1], and a concrete holographic correspondence was given by [2, 3]. It has been intensively studied as a realization of large N duality between gauge theory and string theory. Until 2002, mainly the connections between supergravity modes and BPS operators in SYM had been investigated. However, in paper [6] the correspondence was extended to include certain class of excited string modes and non-BPS operators. In the string theory side, the authors of [6] considered the string states on the pp wave background, which is the Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$. On the other hand in the SYM side, they proposed certain class of operators with large R-charge as the counterpart of the pp wave string states. This is now known as the BMN correspondence and these non-BPS operators are called the BMN operators. Although the BMN correspondence is still between the limited class of string states and Yang-Mills operators, it is a strong support for the full correspondence between these two theories.

On the other hand there is still another interesting connection between SYM and string theory, or precisely, string field theory. This is a correspondence between Wilson loop operator and string field, and a correspondence between the equations of motion of the two. String theory was originally investigated as a theory describing the physics of hadrons, which is now known to be governed by QCD. And there were a lot of works in which they tried to derive the area law of the expectation value of the Wilson loop from its equation of motion (loop equation) [7]. The authors of these papers studied whether the loop equation can be identified with the string equation of Nambu-Goto type. Thus such an idea that there are some relations between the Wilson loop and Nambu-Goto type string is quite an old one. It is an interesting subject to reexamine this old idea from the view point of the AdS/CFT correspondence. In fact, there have been works which discuss the expectation value of the Wilson loop in the context of the AdS/CFT correspondence and the loop equation [8 – 13].

The purpose of this paper is to show that the BMN operators are found in the Wilson loop operator in SYM.[1] Our Wilson loop is the one obtained from "dimensional reduction" of ten-dimensional $\mathcal{N} = 1$ SYM, and it consists of the six scalar fields as well as the gauge field. Therefore, our loop is a "loop in ten-dimensional spacetime". We expand this Wilson loop operator twice. First we expand it in powers of the loop fluctuation around the BPS configuration which is a point in four dimensions and a straight-line in extra six dimensions. Next we further expand each term of the first step in powers of the scalar field Z associated with the straight-line of the BPS configuration. We find that the generic operator in our double expansion is nothing but the BMN operator proposed in [6]. The number of the impurities correspond to the number of the loop fluctuations. Furthermore we will see that even the correction terms introduced and discussed in [15] can be reproduced correctly. In particular, the correction term including the impurity $\bar{Z}$ appears after we impose the locally supersymmetric condition on the whole Wilson loop including the fluctuations. Our finding must be an important step toward the understanding of the relation between the Wilson loop and string field.

This paper is organized as follows. In section 2 we briefly review the BMN correspondence. Section 3 is the main part of this paper. In subsection 3.1 we perform the double expansion we mentioned above. Then we show how the BMN operators proposed in [6] with two or less impurities arise in the expansion. In subsection 3.2 we impose the locally supersymmetric condition on the whole loop, and we show that the BMN operators with correction terms given in [15] also arise in the expansion. Section 4 is devoted to the conclusion and discussion. In appendix A we show that the argument given in subsection 3.1 can be generalized to the case with m impurities, and in appendix B we give an argument which does not need the saddle point approximation used in section 3 and appendix A.

2. BMN correspondence

Before discussing how the BMN operators arise in the Wilson loop, we shall recapitulate the BMN correspondence [6]. This correspondence claims that the anomalous dimension of each BMN operator coincides with the lightcone energy of the corresponding string state

[1] In [14] the BMN correspondence is reproduced by considering the couplings among closed string states having large angular momentum and open string states on D3-branes. This seems to be a limited version of the coupling between a closed string and a Wilson loop [8 – 13]. Thus the observation in [14] seems to be closely related to ours.

string states		BMN operators
$\vert 0\,;p^+\rangle$	$\Longleftrightarrow$	$\mathrm{Tr}[Z^J]$
$a_0^{M\dagger}\vert 0\,;p^+\rangle$	$\Longleftrightarrow$	$\mathrm{Tr}\left[\mathcal{O}_M Z^J\right]$
$a_{n_1}^{M_1\dagger}a_{n_2}^{M_2\dagger}\vert 0\,;p^+\rangle$	$\Longleftrightarrow$	$\sum_{k_2=0}^{J}\mathrm{Tr}\left[\mathcal{O}_{M_1}Z^{k_2}\mathcal{O}_{M_2}Z^{J-k_2}\right]e^{2\pi i n_2 k_2/J}$
$a_{n_1}^{M_1\dagger}\cdots a_{n_m}^{M_m\dagger}\vert 0\,;p^+\rangle$	$\Longleftrightarrow$	$\sum_{\sigma\in S_{m-1}}\sum_{0\le k_{\sigma(2)}\le k_{\sigma(3)}\le\cdots\le k_{\sigma(m)}\le J}\mathrm{Tr}\Big[\mathcal{O}_{M_1}Z^{k_{\sigma(2)}}\mathcal{O}_{M_{\sigma(2)}}Z^{k_{\sigma(3)}-k_{\sigma(2)}}\times\cdots$ $\cdots\times\mathcal{O}_{M_{\sigma(m-1)}}Z^{k_{\sigma(m)}-k_{\sigma(m-1)}}\times$ $\times\,\mathcal{O}_{M_{\sigma(m)}}Z^{J-k_{\sigma(m)}}\Big]e^{2\pi i\sum_{q=2}^{m}n_q k_q/J}$

Table 1: The correspondence between the string states and the BMN operators proposed in [6].

propagating on the pp wave background. The BMN operators are certain class of local operators in $\mathcal{N}=4$ SU(N) SYM in four dimensions, and the pp wave background is realized by taking the Penrose limit of $\mathrm{AdS}_5\times\mathrm{S}^5$. For this reason this correspondence is expected to be a restricted version of the AdS/CFT correspondence.

We summarize in table 1 the BMN correspondence proposed in [6]. On the left hand side (LHS) of the table, $a_n^{M\dagger}$ is the creation operator of a string mode of level n with $M=1,\ldots,8$ specifying the transverse directions in the lightcone gauge. We have chosen a basis of the Fourier modes where $n>0$ ($n<0$) corresponds to the left (right) mover. Each string state on the LHS must satisfy the level matching condition $\sum_{\ell=1}^{m}n_\ell=0$. On the right hand side (RHS) of the table, Z and $\mathcal{O}_M$ ($M=1,\ldots,8$) are defined by $Z=\Phi_5+i\Phi_6$, $\mathcal{O}_\mu=D_\mu Z$ ($\mu=1,\ldots,4$) and $\mathcal{O}_{4+a}=\Phi_a$ ($a=1,\ldots,4$) with Φ_i ($i=1,\ldots,6$) being the six scalar fields in SYM. Each operator on the RHS has a large number $J(\sim\sqrt{N})$ of Z with some finite number of impurities $\mathcal{O}_M$ being inserted with an appropriate phase factor. Summation is taken over the positions of impurities. In particular, the summation over σ is with respect to the permutations of $\{2,3,\ldots,m\}$.

In table 1 we write only terms which contain impurities whose classical conformal dimension Δ minus R-charge $\mathcal{J}$ is equal to 1. In fact it has been known that these expressions need corrections by terms which contain impurities with $\Delta-\mathcal{J}\ge 2$ [15]. The complete forms of the large J limit of the BMN operators corresponding to the string states $a_{-n}^{\mu\dagger}a_n^{\nu\dagger}|0;p^+\rangle$, $a_{-n}^{4+a\dagger}a_n^{\mu\dagger}|0;p^+\rangle$ and $a_{-n}^{4+a\dagger}a_n^{4+b\dagger}|0;p^+\rangle$ are given respectively by

$$\mathcal{O}^J_{\mu\nu,n}=\sum_{k=0}^{J}\mathrm{Tr}\left[D_\mu Z Z^k D_\nu Z Z^{J-k}\right]e^{2\pi i n k/J}+\mathrm{Tr}\left[D_{(\mu}D_{\nu)}ZZ^{J+1}\right], \tag{2.1}$$

$$\mathcal{O}^J_{4+a\,\mu,n}=\sum_{k=0}^{J}\mathrm{Tr}\left[\Phi_a Z^k D_\mu Z Z^{J-k}\right]e^{2\pi i n k/J}+\mathrm{Tr}\left[D_\mu\Phi_a Z^{J+1}\right], \tag{2.2}$$

$$\mathcal{O}^J_{4+a\,4+b,n}=\sum_{k=0}^{J}\mathrm{Tr}\left[\Phi_a Z^k\Phi_b Z^{J-k}\right]e^{2\pi i n k/J}-\frac{1}{2}\delta_{ab}\,\mathrm{Tr}\left[\bar{Z}Z^{J+1}\right], \tag{2.3}$$

where $\bar{Z}=\Phi_5-i\Phi_6$. Other operators on the RHS of table 1 with three or more impurities need similar corrections.

3. BMN operators from the expansion of Wilson loop

In this section we show that the BMN operators arise in the double expansion of the Wilson loop operator. In subsection 3.1 we show how the impurities $\mathcal{O}_M$ $(M = 1, \ldots, 8)$ with $\Delta - \mathcal{J} = 1$ appear with appropriate phase factors. Next, in subsection 3.2 we derive the correction terms with $\Delta - \mathcal{J} = 2$ in (2.1)–(2.3).

3.1 Impurities with $\Delta - \mathcal{J} = 1$ and their phase factors

Let us consider the Wilson loop operator consisting of the scalar fields Φ_i as well as the gauge fields A_μ [8 – 13]:

$$W(C) = \mathrm{Tr}\left[\mathrm{P}\exp\left(\int_0^t ds\Big(A_\mu(x(s))\dot{x}^\mu(s) + \Phi_i(x(s))\dot{y}^i(s)\Big)\right)\right] = \mathrm{Tr}\left[\mathrm{P}w_0^t(C)\right]. \tag{3.1}$$

The existence of the scalar fields in the Wilson loop looks natural if we recall that the four-dimensional $\mathcal{N} = 4$ SYM is obtained as a dimensional reduction of ten-dimensional $\mathcal{N} = 1$ SYM. The scalar fields are essential for reproducing the BMN operators and also for deriving the loop equations (see section 4).[2] As the loop C we take $C = C_0 + \delta C$ with a BPS "loop" configuration C_0 given by

$$x^\mu_{C_0}(s) = x^\mu\,, \qquad \dot{y}^i_{C_0}(s) = (0, 0, 0, 0, 1, i)\,, \tag{3.2}$$

and a small fluctuation $\delta C = \{\delta x^\mu(s), \delta y^i(s)\}$ around it. The Wilson loop (3.1) is locally supersymmetric if the condition

$$\dot{x}(s)^2 - \dot{y}(s)^2 = 0 \tag{3.3}$$

is satisfied and is 1/2 BPS if the loop is a straight-line [10, 11]. Note that $W(C)$ is a functional of $x^\mu(s)$ and $\dot{y}^i(s)$, and hence we take $x^\mu(s)$ and $\dot{y}^i(s)$ to be periodic functions of s with period t. This means that our Wilson "loop" is not necessarily closed in the direction of y^i, but this dose not spoil the gauge invariance of this operator. Therefore, we take $\delta x^\mu(s)$ and $\delta\dot{y}^i(s)$ as periodic functions:

$$\delta x^\mu(s) = \sum_{n=-\infty}^{\infty} \delta x^\mu_n e^{2\pi i n s/t}\,, \qquad \delta\dot{y}^i(s) = \sum_{n=-\infty}^{\infty} \delta\dot{y}^i_n e^{2\pi i n s/t}\,. \tag{3.4}$$

Note in particular that $\dot{y}^i$ and $\delta\dot{y}^i$ have zero modes. We Taylor-expand the Wilson loop $W(C) = W(C_0 + \delta C)$ around C_0.[3] Since $W(C)$ is a functional of $x^\mu(s)$ and $\dot{y}^i(s)$, we have

[2] For the same reason, the fermionic extension of the Wilson loop is also necessary. However, in this paper, we concentrate only on the bosonic part.

[3] Similar expansion is also considered in [16] in the context of the connection between the open Wilson line in non-commutative Yang-Mills theory and the closed string field. In [12], the expansion of the Wilson loop in terms of the local operators is discussed in the context of the AdS/CFT and the BMN correspondence.

JHEP06(2005)050

$$
\begin{aligned}
W(C) &= W(C_0) + \int_0^t ds\, \delta x^\mu(s) \frac{\delta W(C)}{\delta x^\mu(s)}\bigg|_{C=C_0} + \int_0^t ds\, \delta \dot{y}^i(s) \frac{\delta W(C)}{\delta \dot{y}^i(s)}\bigg|_{C=C_0} + \\
&\quad + \frac{1}{2}\int_0^t ds_1 \int_0^t ds_2\, \delta x^\mu(s_1)\delta x^\nu(s_2) \frac{\delta^2 W(C)}{\delta x^\mu(s_1)\delta x^\nu(s_2)}\bigg|_{C=C_0} + \cdots \\
&= W(C_0) + \sum_n \delta x_n^\mu \int_0^t ds \frac{\delta W(C)}{\delta x^\mu(s)}\bigg|_{C=C_0} e^{2\pi i n s/t} + \sum_n \delta \dot{y}_n^i \int_0^t ds \frac{\delta W(C)}{\delta \dot{y}^i(s)}\bigg|_{C=C_0} e^{2\pi i n s/t} + \\
&\quad + \frac{1}{2}\sum_{n_1,n_2} \delta x_{n_1}^\mu \delta x_{n_2}^\nu \int_0^t ds_1 \int_0^t ds_2 \frac{\delta^2 W(C)}{\delta x^\mu(s_1)\delta x^\nu(s_2)}\bigg|_{C=C_0} e^{2\pi i (n_1 s_1 + n_2 s_2)/t} + \cdots . \qquad (3.5)
\end{aligned}
$$

Let us look at each term in this expansion in detail. The first term $W(C_0)$ is given by

$$
W(C_0) = \mathrm{Tr}\left[\exp\left(\int_0^t ds Z(x)\right)\right] = \sum_{J=0}^{\infty} \frac{t^J}{J!}\,\mathrm{Tr}\left[Z^J\right](x)\,, \qquad (3.6)
$$

where we have used that $\dot{x}^\mu(s) = 0$ and $\Phi_i(x(s))\dot{y}^i(s) = Z(x)$ on the path C_0. Note that there appear the BMN operator $\mathrm{Tr}\left[Z^J\right]$ corresponding to the ground state of a pp wave string (see table 1). Next we examine the terms which contain one functional derivative. For this, we use the following formula of functional derivatives:

$$
\begin{aligned}
\frac{\delta W(C)}{\delta x^\mu(s)}\bigg|_{C=C_0} &= \mathrm{Tr}\left[\left(F_{\mu\nu}(x(s))\dot{x}^\nu(s) + D_\mu \Phi_i(x(s))\dot{y}^i(s)\right) \mathrm{P}\left(w_s^{s+t}(C)\right)\right]\Big|_{C=C_0} \\
&= \mathrm{Tr}\left[D_\mu Z e^{tZ}\right](x)\,, \qquad (3.7) \\
\frac{\delta W(C)}{\delta \dot{y}^i(s)}\bigg|_{C=C_0} &= \mathrm{Tr}\left[\Phi_i e^{tZ}\right](x)\,, \qquad (3.8)
\end{aligned}
$$

where $w_s^{s'}(C)$ represents, as in eq. (3.1), the Wilson line along C from s to s' without trace. Then we have

$$
\begin{aligned}
\sum_n \delta x_n^\mu \int_0^t ds \frac{\delta W(C)}{\delta x^\mu(s)}\bigg|_{C=C_0} e^{2\pi i n s/t} &= \sum_n \delta x_n^\mu \int_0^t ds\, \mathrm{Tr}\left[D_\mu Z e^{tZ}\right](x) e^{2\pi i n s/t} \\
&= \delta x_0^\mu \sum_{J=0}^{\infty} \frac{t^{J+1}}{J!}\,\mathrm{Tr}\left[D_\mu Z Z^J\right](x)\,. \qquad (3.9)
\end{aligned}
$$

The s-integration yields the Kronecker delta $t\delta_{n,0}$, which picks up only the zero mode, namely the SUGRA mode, from the summation over n. The SUGRA modes in the other directions are supplied from functional derivatives with respect to $\dot{y}^i(s)$:

$$
\sum_n \delta \dot{y}_n^i \int_0^t ds \frac{\delta W(C)}{\delta \dot{y}^i(s)}\bigg|_{C=C_0} e^{2\pi i n s/t} = \delta \dot{y}_0^i \sum_{J=0}^{\infty} \frac{t^{J+1}}{J!}\,\mathrm{Tr}\left[\Phi_i Z^J\right](x)\,. \qquad (3.10)
$$

The SUGRA modes correspond to $i = 1, \ldots, 4$, while the meaning of the $i = 5, 6$ terms is not clear at the present stage. Here we simply ignore the $i = 5, 6$ terms. However, we shall see in subsection 3.2 that these terms play important roles in reproducing the correction terms. From (3.9) and (3.10), we expect the following correspondence between the functional derivatives and the oscillation modes of the pp wave string (see table 1):

$$\int_0^t ds \frac{\delta}{\delta x^\mu(s)} \longrightarrow a_0^{\mu\dagger}\,, \qquad \int_0^t ds \frac{\delta}{\delta \dot{y}^a(s)} \longrightarrow a_0^{4+a\dagger}\,. \tag{3.11}$$

Next we shall study whether higher string modes can be derived in the same manner. For this purpose we consider the terms which contain two functional derivatives in the expansion (3.5) of the Wilson loop. Here we use the dilute gas approximation and assume that the parameters s_1 and s_2 of the two functional derivatives are not close to each other. In particular, we omit the contribution from the terms proportional to $\delta(s_1 - s_2)$ and $\partial_{s_1}\delta(s_1 - s_2)$. We will examine these terms in the next subsection. With this approximation, the following replacement formulas similar to (3.7) and (3.8) hold:

$$\frac{\delta}{\delta x^\mu(s)} \rightarrow D_\mu Z(x(s))\,, \qquad \frac{\delta}{\delta \dot{y}^i(s)} \rightarrow \Phi_i(x(s))\,. \tag{3.12}$$

Thus we must consider terms of the following form:

$$\begin{aligned}\int_0^t ds_1 \int_{s_1}^{s_1+t} ds_2 \,\mathrm{Tr}\left[\mathcal{O}_{M_1}(x) w_{s_1}^{s_2}(C_0) \mathcal{O}_{M_2}(x) w_{s_2}^{s_1+t}(C_0)\right] e^{2\pi i(n_1 s_1 + n_2 s_2)/t} = \\ = \int_0^t ds_1 \int_{s_1}^{s_1+t} ds_2 \sum_{J=0}^{\infty} \sum_{k_2=0}^{J} \frac{1}{(J-k_2)!k_2!}(s_2 - s_1)^{k_2}(s_1 + t - s_2)^{J-k_2} \times \\ \times \,\mathrm{Tr}\left[\mathcal{O}_{M_1} Z^{k_2} \mathcal{O}_{M_2} Z^{J-k_2}\right](x)\, e^{2\pi i(n_1 s_1 + n_2 s_2)/t}\,.\end{aligned} \tag{3.13}$$

By adopting the new integration variables $(\tilde{s}_1, \tilde{s}_2) = (s_1, s_2 - s_1)/t$, the integration over $\tilde{s}_1$ can be performed to yield the Kronecker delta $t\delta_{n_1+n_2,0}$, and (3.13) is further rewritten into

$$\delta_{n_1+n_2,0} \sum_{J=0}^{\infty} t^{J+2} \sum_{k_2=0}^{J} \mathrm{Tr}\left[\mathcal{O}_{M_1} Z^{k_2} \mathcal{O}_{M_2} Z^{J-k_2}\right](x) F_2(n_2, k_2, J)\,, \tag{3.14}$$

where we have defined

$$F_2(n_2, k_2, J) \equiv \frac{1}{(J-k_2)!k_2!} \int_0^1 d\tilde{s}_2 \tilde{s}_2^{k_2} (1 - \tilde{s}_2)^{J-k_2} e^{2\pi i n_2 \tilde{s}_2}\,. \tag{3.15}$$

In the large J limit, the integration over $\tilde{s}_2$ can be evaluated using the saddle point approximation to give[4]

$$F_2(n_2, k_2, J) \sim \frac{1}{JJ!} \exp\left(\frac{2\pi i n_2 k_2}{J}\right), \tag{3.16}$$

where we have used Stirling's formula, i.e., $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$ $(n \gg 1)$. The saddle point

[4]For the validity of the saddle point approximation, it is necessary that both k_2/J and $1 - (k_2/J)$ are of $O(1)$ as can be seen from the coefficients of the non-gaussian terms of the fluctuation around the saddle point. Since the saddle point is at $\tilde{s}_2 = k_2/J$, this condition is consistent with the dilute gas assumption.

JHEP06(2005)050

of the integration (3.15) is at $\tilde{s}_2 = k_2/J$. Finally we can evaluate the operators in the J-summation of (3.14) for large J limit as

$$t^{J+2}\frac{\delta_{n_1+n_2,0}}{JJ!}\sum_{k_2=0}^{J}\mathrm{Tr}\left[\mathcal{O}_{M_1}Z^{k_2}\mathcal{O}_{M_2}Z^{J-k_2}\right](x)e^{2\pi i n_2 k_2/J}\,. \tag{3.17}$$

This is nothing but the BMN operator with two impurities given in table 1. As seen from the above derivation, the expression (3.17) may be incorrect in the region where k_2/J is close to 0 or 1, namely, when the two impurities $\mathcal{O}_{M_1}$ and $\mathcal{O}_{M_2}$ are close to each other. However, we shall show in appendix B that (3.17) is in fact valid for all k_2 so long as J is large.

3.2 Complete forms of the large J BMN operators

In this subsection we show that the BMN operators with the correction terms, (2.1)–(2.3), arise from the double expansion of the Wilson loop (3.5). The origins of the correction terms are i) two functional derivatives acting at the same point on the loop which we neglected in the previous subsection, and ii) the functional derivatives $\delta/\delta\dot{y}^i$ with $i=5,6$ together with the locally supersymmetric constraint on the fluctuation of the loop which we newly impose in this subsection.

First, two functional derivatives acting on $W(C)$ is given, including the delta function terms, by

$$\begin{aligned}\left.\frac{\delta^2 W(C)}{\delta x^\mu(s_1)\delta x^\nu(s_2)}\right|_{C=C_0} &= \theta(s_2-s_1)\,\mathrm{Tr}\left[w_0^{s_1}(C_0)D_\mu Z(x)w_{s_1}^{s_2}(C_0)D_\nu Z(x)w_{s_2}^{t}(C_0)\right]+\\ &\quad+\theta(s_1-s_2)\,\mathrm{Tr}\left[w_0^{s_2}(C_0)D_\nu Z(x)w_{s_2}^{s_1}(C_0)D_\mu Z(x)w_{s_1}^{t}(C_0)\right]+\\ &\quad+\delta(s_1-s_2)\,\mathrm{Tr}\left[w_0^{s_1}(C_0)D_{(\mu}D_{\nu)}Z(x)w_{s_1}^{t}(C_0)\right]+\\ &\quad+\frac{1}{2}\partial_{s_2}\delta(s_1-s_2)\,\mathrm{Tr}\big[w_0^{s_1}(C_0)F_{\mu\nu}(x(s_1))w_{s_1}^{t}(C_0)+\\ &\qquad\qquad+w_0^{s_2}(C_0)F_{\mu\nu}(x(s_2))w_{s_2}^{t}(C_0)\big],\end{aligned} \tag{3.18}$$

$$\begin{aligned}\left.\frac{\delta^2 W(C)}{\delta \dot{y}^i(s_1)\delta x^\mu(s_2)}\right|_{C=C_0} &= \theta(s_2-s_1)\,\mathrm{Tr}\left[w_0^{s_1}(C_0)\Phi_i(x)w_{s_1}^{s_2}(C_0)D_\mu Z(x)w_{s_2}^{t}(C_0)\right]+\\ &\quad+\theta(s_1-s_2)\,\mathrm{Tr}\left[w_0^{s_2}(C_0)D_\mu Z(x)w_{s_2}^{s_1}(C_0)\Phi_i(x)w_{s_1}^{t}(C_0)\right]+\\ &\quad+\delta(s_1-s_2)\,\mathrm{Tr}\left[w_0^{s_1}(C_0)D_\mu\Phi_i(x)w_{s_1}^{t}(C_0)\right],\end{aligned} \tag{3.19}$$

$$\begin{aligned}\left.\frac{\delta^2 W(C)}{\delta \dot{y}^i(s_1)\delta \dot{y}^j(s_2)}\right|_{C=C_0} &= \theta(s_2-s_1)\,\mathrm{Tr}\left[w_0^{s_1}(C_0)\Phi_i(x)w_{s_1}^{s_2}(C_0)\Phi_j(x)w_{s_2}^{t}(C_0)\right]+\\ &\quad+\theta(s_1-s_2)\,\mathrm{Tr}\left[w_0^{s_2}(C_0)\Phi_j(x)w_{s_2}^{s_1}(C_0)\Phi_i(x)w_{s_1}^{t}(C_0)\right],\end{aligned} \tag{3.20}$$

with $\theta(s)=1$ $(=0)$ for $s>0$ $(s<0)$, and $D_{(\mu}D_{\nu)} \equiv (D_\mu D_\nu + D_\nu D_\mu)/2$. We have used the Bianchi identity of $F_{\mu\nu}$ in obtaining (3.18) which is manifestly symmetric under the exchange $(s_1,\mu)\leftrightarrow(s_2,\nu)$. In the previous subsection, we omitted terms proportional to $\delta(s_1-s_2)$ and $\delta'(s_1-s_2)$. Using (3.18)–(3.20), the terms with two functional derivatives in (3.5) are reduced to

$$
\begin{aligned}
&\int_0^t ds_1 \int_0^t ds_2 \delta x^\mu(s_1)\delta x^\nu(s_2) \frac{\delta^2 W(C)}{\delta x^\mu(s_1)\delta x^\nu(s_2)}\bigg|_{C=C_0} = \\
&\qquad = \sum_J \sum_{n=-\infty}^{\infty} t^{J+2}\delta x^\mu_{-n}\delta x^\nu_n \Bigg\{ \sum_{k=0}^{J} \mathrm{Tr}\left[D_\mu Z Z^k D_\nu Z Z^{J-k}\right] F_2(n,k,J) + \\
&\qquad\qquad + \frac{1}{(J+1)!}\mathrm{Tr}\left[D_{(\mu}D_{\nu)} Z Z^{J+1}\right] - \\
&\qquad\qquad - \frac{2\pi i n}{(J+2)!}\mathrm{Tr}\left[F_{\mu\nu} Z^{J+2}\right] \Bigg\} \\
&\qquad \sim \sum_J \frac{1}{JJ!} \sum_{n=-\infty}^{\infty} t^{J+2}\delta x^\mu_{-n}\delta x^\nu_n \mathcal{O}^J_{\mu\nu,n}\,,
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
&\int_0^t ds_1 \int_0^t ds_2 \delta \dot{y}^a(s_1)\delta x^\mu(s_2) \frac{\delta^2 W(C)}{\delta \dot{y}^a(s_1)\delta x^\mu(s_2)}\bigg|_{C=C_0} = \\
&\qquad = \sum_J \sum_{n=-\infty}^{\infty} t^{J+2}\delta \dot{y}^a_{-n}\delta x^\mu_n \Bigg\{ \sum_{k=0}^{J} \mathrm{Tr}\left[\Phi_a Z^k D_\mu Z Z^{J-k}\right] F_2(n,k,J) + \\
&\qquad\qquad + \frac{1}{(J+1)!}\mathrm{Tr}\left[D_\mu \Phi_a Z^{J+1}\right] \Bigg\} \\
&\qquad \sim \sum_J \frac{1}{JJ!} \sum_{n=-\infty}^{\infty} t^{J+2}\delta \dot{y}^a_{-n}\delta x^\mu_n \mathcal{O}^J_{4+a\,\mu,n}\,,
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
&\int_0^t ds_1 \int_0^t ds_2 \delta \dot{y}^a(s_1)\delta \dot{y}^b(s_2) \frac{\delta^2 W(C)}{\delta \dot{y}^a(s_1)\delta \dot{y}^b(s_2)}\bigg|_{C=C_0} = \\
&\qquad = \sum_J \sum_{n=-\infty}^{\infty} t^{J+2}\delta \dot{y}^a_{-n}\delta \dot{y}^b_n \sum_{k=0}^{J} \mathrm{Tr}\left[\Phi_a Z^k \Phi_b Z^{J-k}\right] F_2(n,k,J) \\
&\qquad \sim \sum_J \frac{1}{JJ!} \sum_{n=-\infty}^{\infty} t^{J+2}\delta \dot{y}^a_{-n}\delta \dot{y}^b_n \sum_{k=0}^{J} \mathrm{Tr}\left[\Phi_a Z^k \Phi_b Z^{J-k}\right] e^{2\pi i n k/J}\,.
\end{aligned}
\tag{3.23}
$$

In the last line of each equation we have taken the large J limit using (3.16). Furthermore, in (3.21), we have neglected the last term containing $F_{\mu\nu}$ in the second expression, because this term is multiplied by an additional factor $1/J$ compared to the other two terms. In (3.22) and (3.23), the indices a and b run from 1 to 4, and we have omitted the fluctuations $\delta\dot{y}^5$ and $\delta\dot{y}^6$. We will see later that $\delta\dot{y}^5$ and $\delta\dot{y}^6$ contribute to the terms with more powers of the transverse fluctuations. In (3.21) and (3.22) we have obtained the correct operators $\mathcal{O}^J_{\mu\nu,n}$ (2.1) and $\mathcal{O}^J_{4+a\,\mu,n}$ (2.2), respectively. On the other hand, the operator in the last expression of (3.23) differs from $\mathcal{O}^J_{4+a\,4+b,n}$ (2.3) in that the former lacks the term $-\delta_{ab}\,\mathrm{Tr}[\bar{Z}Z^{J+1}]/2$. This is because the four directions $\delta\dot{y}^a$ ($a = 1,\ldots,4$) do not directly correspond to the transverse directions of the pp wave string modes in the lightcone gauge. In the following, we shall show that the correct choice of the four coordinates can be found by imposing the locally supersymmetric condition (3.3) on the whole loop including the fluctuations.[5]

[5] In [9], this locally supersymmetric condition is imposed on the Wilson loop operator from the start.

Before proceeding to discuss the locally supersymmetric condition, we shall give some comments on our double expansion. Introducing the new coordinates $z(s) \equiv (y^5(s) + iy^6(s))/2$ and $\bar{z}(s) \equiv (y^5(s) - iy^6(s))/2$, we have $\sum_{i=5,6} \Phi_i \dot{y}^i = Z\dot{\bar{z}} + \bar{Z}\dot{z}$, and C_0 (3.2) is rewritten as

$$x^\mu_{C_0}(s) = x^\mu \,, \qquad \dot{y}^a_{C_0}(s) = \dot{z}_{C_0}(s) = 0 \,, \qquad \dot{\bar{z}}_{C_0}(s) = 1 \,. \tag{3.24}$$

Using the reparametrization invariance, we can fix $\dot{\bar{z}}(s) = \dot{\bar{z}}_{C_0}(s) + \delta\dot{\bar{z}}(s)$ equal to its value of C_0:

$$\dot{\bar{z}}(s) = 1 \,. \tag{3.25}$$

In other words, we can always gauge away the small fluctuation $\delta\dot{\bar{z}}(s)$, i.e., $\delta\dot{\bar{z}}(s) = 0$. Using this gauge, our double expansion is reduced to the following two steps. First we expand around the straight-line C_0 with respect to the nine coordinates $\delta x^\mu(s)$, $\delta\dot{y}^a(s)$ and $\delta\dot{z}(s)$. Next we expand this straight-line itself with respect to Z.

Now let us impose the locally supersymmetric condition (3.3) on the whole loop including the fluctuation:

$$\left(\dot{x}^\mu_{C_0}(s) + \delta\dot{x}^\mu(s)\right)^2 - \left(\dot{y}^i_{C_0}(s) + \delta\dot{y}^i(s)\right)^2 = \left(\delta\dot{x}^\mu(s)\right)^2 - 4\dot{\bar{z}}(s)\delta\dot{z}(s) - \left(\delta\dot{y}^a(s)\right)^2 = 0 \,. \tag{3.26}$$

Using the gauge condition (3.25) we can solve this constraint for $\delta\dot{z}(s)$:

$$\delta\dot{z}(s) = \frac{1}{4}\left(\left(\delta\dot{x}^\mu(s)\right)^2 - \left(\delta\dot{y}^a(s)\right)^2\right) . \tag{3.27}$$

This constraint does supply the missing correction term for (3.23) mentioned above. To see this we reexamine the terms containing one functional derivative with respect to $\dot{y}^5(s)$ and $\dot{y}^6(s)$ in (3.5):

$$\int_0^t ds \left(\delta\dot{y}^5(s)\frac{\delta W(C)}{\delta\dot{y}^5(s)} + \delta\dot{y}^6(s)\frac{\delta W(C)}{\delta\dot{y}^6(s)}\right)\bigg|_{C=C_0} = \int_0^t ds\delta\dot{z}(s)\frac{\delta W(C)}{\delta\dot{z}(s)}\bigg|_{C=C_0} , \tag{3.28}$$

where we have used the gauge condition $\delta\dot{\bar{z}}(s) = 0$. Using

$$\frac{\delta W(C)}{\delta\dot{z}(s)}\bigg|_{C=C_0} = \sum_{J=0}^{\infty}\frac{t^J}{J!}\,\mathrm{Tr}\left[\bar{Z}Z^J\right](x) \,, \tag{3.29}$$

and the locally supersymmetric condition (3.27), we obtain

$$\int_0^t ds\delta\dot{z}(s)\frac{\delta W(C)}{\delta\dot{z}(s)}\bigg|_{C=C_0} = \frac{1}{4}\sum_{J=0}^{\infty}\frac{1}{J!}\sum_{n=-\infty}^{\infty}\left((2\pi n)^2 t^{J-1}\delta x^\mu_{-n}\delta x^\mu_n - t^{J+1}\delta\dot{y}^a_{-n}\delta\dot{y}^a_n\right)\mathrm{Tr}[\bar{Z}Z^J](x) \,, \tag{3.30}$$

which are quadratic in the fluctuations and should be added to (3.21) and (3.23). For (3.21), the $(\delta x)^2$ term of (3.30) adds $\delta_{\mu\nu}(2\pi n)^2\,\mathrm{Tr}\,\bar{Z}Z^{J+3}/2(J+3)!$ to the second line of (3.21) (here we have adjusted the conformal dimension and the R-charge of the added operator). However, the added operator has an extra factor $1/J^2$ compared to $\mathcal{O}^J_{\mu\nu,n}$ and hence we can neglect it. On the other hand, we should add $-\delta_{ab}\,\mathrm{Tr}\,\bar{Z}Z^{J+1}/2(J+1)!$ to the last line of (3.23). This term is exactly what is needed in order to have the correct BMN operator $\mathcal{O}^J_{4+a\,4+b,n}$. Thus the locally supersymmetric condition (3.27) extracts the correct fluctuations corresponding to the transverse directions of the pp wave string. In the same manner as we have shown in (3.28)–(3.30), the terms containing functional derivatives with respect to $\dot{y}^5(s)$ and $\dot{y}^6(s)$ in (3.19) and (3.20) contribute to the string states with three or more oscillators.

In this section we have shown that the BMN operators corresponding to the string states with two or less excited modes arise in the Taylor expansion of the Wilson loop operator (3.5). In subsection 3.1, neglecting the contribution from two functional derivatives acting at the same point, we derived the correspondence between the fluctuation modes of the string and those of the Wilson loop. In appendix A we generalize this argument to the BMN operators with an arbitrary number of impurities. This observation strongly suggests that we can reinterpret the BMN correspondence, and moreover the full AdS/CFT correspondence, as the relation between a string and a Wilson loop. Further discussion including the relation between the equations of motion of these dynamical variables will be given in the next concluding section. In subsection 3.2 we have shown that the correction terms in (2.1)–(2.3) are also found in the expansion (3.5). The correction terms in (2.1) and (2.2) arise from two functional derivatives acting at the same point which we neglected in subsection 3.1. However, as for (2.3), the existence of the correction term is the result of the mismatch between the two sets of four directions $\{\delta y^a_n\}$ and $\{a^{4+a\dagger}_n\}$. Imposing the locally supersymmetric condition (3.27) we have extracted the correct four directions and reproduced the correction term in (2.3) from the expansion of the Wilson loop.

Finally we give the large J part of the expansion of $W(C)$:

$$\begin{aligned} W(C) \sim \sum_J \frac{t^J}{J!}\Bigg\{ & \mathcal{O}^J_{\text{ground}} + t\left(\delta x^\mu_0 \mathcal{O}^J_{\mu,0} + \delta\dot{y}^a_0 \mathcal{O}^J_{4+a,0}\right) + \\ & + \frac{t}{2!}\Bigg(\sum_n \delta x^\mu_{-n}\delta x^\nu_n \mathcal{O}^{J-1}_{\mu\nu,n} + \sum_n \delta\dot{y}^a_{-n}\delta x^\mu_n \mathcal{O}^{J-1}_{4+a\,\mu,n} + \\ & + \sum_n \delta\dot{y}^a_{-n}\delta\dot{y}^b_n \mathcal{O}^{J-1}_{4+a\,4+b,n}\Bigg) + \cdots\Bigg\}. \end{aligned} \tag{3.31}$$

with $\mathcal{O}^J_{\text{ground}} = \mathrm{Tr}[Z^J]$, $\mathcal{O}^J_{\mu,0} = \mathrm{Tr}[D_\mu Z Z^J]$ and $\mathcal{O}^J_{4+a,0} = \mathrm{Tr}[\Phi_a Z^J]$. The dots represents terms with three or more fluctuations.

4. Conclusion and discussion

In this paper we clarified how the BMN operators, including the correction terms, are embedded in the Wilson loop operator in four-dimensional $\mathcal{N}=4$ SYM. First we expanded the Wilson loop in powers of the fluctuations around the BPS configuration (3.2). Then we further expanded each term in this series in powers of the scalar field Z. We saw that the operators with large number of Z in this expansion are nothing but the BMN operators. The nontrivial phase factors in the BMN operators appear in a natural way from the mode functions $e^{2\pi i n s/t}$ for the fluctuations of the loop, (3.4). For the generic configurations of the functional derivatives, the number of the impurities in the BMN operator coincides with the power of the fluctuations, which is equal to the number of functional derivatives operating the Wilson loop. Using this identification we can reinterpret the BMN correspondence as the relation between the fluctuation modes of the Wilson loop and those of the string. This correspondence is summarized in table 2. We have also shown that, in order to derive the correction terms of the large J BMN operators, naive identification of the directions (μ, a) such as the one given in table 2 without any constraint is not correct. This is because there is some mismatch between the directions of $\{\delta\dot{y}^a_n\}$ and $\{a^{4+a\dagger}_n\}$. We must extract the correct four directions from $\{\delta\dot{y}^i\}$ corresponding to the transverse directions of the pp wave string modes in the lightcone gauge. This extraction was realized by imposing the locally supersymmetric condition (3.27).

SYM side		string side
$W(C_0)$	$\longrightarrow$	$\vert 0; p^+\rangle$
$\int_0^t ds \frac{\delta}{\delta x^\mu(s)} e^{2\pi ins/t}$	$\longrightarrow$	$a^{\mu\dagger}_n$
$\int_0^t ds \frac{\delta}{\delta \dot{y}^a(s)} e^{2\pi ins/t}$	$\longrightarrow$	$a^{4+a\dagger}_n$

Table 2: The map from the functional derivatives to the excited string modes. The locally supersymmetric condition is necessary to give the correction terms.

The correspondence in table 2 connects only the narrow sectors in both of the two theories, SYM and string theory. In the string side the spacetime geometry is the pp wave which is the Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$. On the other hand, in the SYM side the corresponding local operators in the expansion of the Wilson loop are limited to those containing a large number of scalar fields Z. However, we expect that there is a full correspondence between the Wilson loop operator in four-dimensional $\mathcal{N} = 4$ SYM and the type IIB superstring field on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. Furthermore we expect that there is a correspondence between the equations of motion which these dynamical variables obey. In the following, we shall discuss these points more concretely.

First consider the expansion of the Wilson loop in powers of the fluctuations:

$$W(C) = \sum \mathcal{O}_{\{M,n\}}(x)\delta X^{M_1}_{n_1}\delta X^{M_2}_{n_2}\cdots\delta X^{M_m}_{n_m}\,, \tag{4.1}$$

where δX is one of the fluctuations $\{\delta x^\mu, \delta y^i\}$, and the operators $\mathcal{O}_{\{M,n\}}(x)$ are written in terms of local fields in SYM. On the other hand, the string field $\Psi[X = X_0 + \delta X]$ which is a functional of the fluctuation $\delta X(\sigma)$ around the zero-mode X_0 can be expanded in terms of spacetime component fields as

$$\Psi[X(\sigma)] = \sum \psi_{\{M,n\}}(X_0)\big\langle\delta X\big|a^{M_1\dagger}_{n_1}a^{M_2\dagger}_{n_2}\cdots a^{M_m\dagger}_{n_m}\big|\text{ground state}\big\rangle\,, \tag{4.2}$$

where $a^{M\dagger}_n$ are the creation operators in the Hilbert space of the first quantized string, and $\psi_{\{M,n\}}(X_0)$ are the component fields. Note that $\big\langle\delta X\big|a^{M_1\dagger}_{n_1}a^{M_2\dagger}_{n_2}\cdots a^{M_m\dagger}_{n_m}\big|\text{ground state}\big\rangle$ on the RHS of (4.2), namely, the wave function of the string state in the δX^M_n-diagonal representation, is roughly equal to $\delta X^{M_1}_{n_1}\delta X^{M_2}_{n_2}\cdots\delta X^{M_m}_{n_m}$ on the RHS of (4.1) up to a gaussian like function of δX^M_n. Now we mean, by the correspondence between the Wilson loop and the string field, that there exists a complete map between the operators and the component fields:

$$\mathcal{O}_{\{M,n\}}(x) \iff \psi_{\{M,n\}}(X_0)\,. \tag{4.3}$$

Our analysis section 3 supports this correspondence in the pp wave limit, and we expect that the full $\mathrm{AdS}_5 \times \mathrm{S}^5$ version of the correspondence does exist.

Next let us discuss the equations of motion. The equation of motion of string field on the curved geometry G_{MN} should be given by

$$\int d\sigma \left\{ -G^{MN}(X)\frac{\delta^2}{\delta X^M \delta X^N} + G_{MN}(X)X^{M\prime}X^{N\prime} \right\}(\sigma)\Psi[X] + (\text{interaction terms}) = 0\,, \tag{4.4}$$

and we expect that the Wilson loop $W(C)$ (by multiplied the gaussian like function mentioned above) satisfies the same equation as (4.4). In fact, it has been known that the Wilson loop operator is subject to the loop equation which resembles (4.4) [7].[6] The starting point of the loop equation is the following formula for the second functional derivative of $W(C)$:

$$\frac{\delta^2}{\delta X^M(s_1)\delta X^N(s_2)}W(C) = X^{P\prime}(s_1)X^{Q\prime}(s_2)\,\mathrm{Tr}\left[\mathrm{P}\left(F_{MP}\big(x(s_1)\big)F_{NQ}\big(x(s_2)\big)w_0^t(C)\right)\right] + \\ + \big(\text{terms with } \delta(s_1 - s_2) \;\text{ or }\; \delta'(s_1 - s_2)\big). \tag{4.5}$$

We want to derive from (4.5) an equation for $W(C)$ which is equivalent to (4.4). This would be possible if the equation like

$$G^{MN}(X(s))F_{MP}(X(s))F_{NQ}(X(s)) \sim G_{PQ}(X(s))\,, \tag{4.6}$$

is realized in some sense for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric (a modification due to the gaussian like function is necessary). The last delta function terms of (4.5) is expected to contribute the interaction terms. Further investigation on this subject is necessary.

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A. Multi-impurities

In this appendix we present the calculation of the general terms in the Taylor expansion of the Wilson loop operator. Neglecting the contribution from the coincident s configurations,

[6] An interesting connection between the IIB matrix model and the string field theory is also discussed in [17].

the term containing m functional derivatives is given by the sum of the following type of operators over the permutations of $\{\mathcal{O}_{M_2}, \mathcal{O}_{M_3}, \cdots, \mathcal{O}_{M_m}\}$:

$$\int_0^t ds_1 \int_{s_1}^{s_1+t} ds_2 \cdots \int_{s_{m-1}}^{s_1+t} ds_m \,\mathrm{Tr}\left[\mathcal{O}_{M_1}(x) w_{s_1}^{s_2}(C_0) \mathcal{O}_{M_2}(x) w_{s_2}^{s_3}(C_0) \cdots \right.$$
$$\left. \cdots w_{s_{m-1}}^{s_m}(C_0) \mathcal{O}_{M_m}(x) w_{s_m}^{s_1+t}(C_0)\right] e^{2\pi i \sum_{q=1}^m n_q s_q/t}. \quad \text{(A.1)}$$

Repeating the same steps as in section 3, we can rewrite it as

$$\delta_{\sum_{q=1}^m n_q, 0} \sum_{J=0}^{\infty} \sum_{0 \le k_2 \le k_3 \le \cdots \le k_m \le J} t^{J+m} \,\mathrm{Tr}\left[\mathcal{O}_{M_1} Z^{k_2} \mathcal{O}_{M_2} Z^{k_3-k_2} \cdots Z^{k_m-k_{m-1}} \mathcal{O}_{M_m} Z^{J-k_m}\right](x) \times$$
$$\times F_m\big(\{n_q\}, \{k_q\}, J\big), \quad \text{(A.2)}$$

where F_m is defined by

$$F_m\big(\{n_q\}, \{k_q\}, J\big) \equiv \frac{1}{k_2!(k_3-k_2)! \cdots (k_m-k_{m-1})!(J-k_m)!} \times$$
$$\times \left(\prod_{q=2}^m \int_0^{1-\sum_{r=2}^{q-1} \tilde{s}_r} d\tilde{s}_q \tilde{s}_q^{k_q-k_{q-1}} \exp\left(2\pi i \sum_{r=q}^m n_r \tilde{s}_q\right)\right) \left(1 - \sum_{r=2}^m \tilde{s}_r\right)^{J-k_m}, \quad \text{(A.3)}$$

with $k_1 \equiv 0$ and $\sum_{r=2}^1 \tilde{s}_r = 0$. The new integration variables with tilde are $\tilde{s}_q = \big(s_q - s_{q-1}\big)/t$ $(q = 2, \ldots, m)$. We carry out the $\tilde{s}$-integrations in the large $k_q - k_{q-1}$ and $J - k_m$ limit by using the saddle point approximation as we did in section 3. The saddle point is at $\tilde{s}_q = (k_q - k_{q-1})/J$, and we obtain

$$F_m\big(\{n_q\}, \{k_q\}, J\big) \sim \frac{1}{J^{m-1} J!} \exp\left(\frac{2\pi i \sum_{q=2}^m n_q k_q}{J}\right). \quad \text{(A.4)}$$

As we noted in section 3, this expression is valid only in the region of k_q where none of $(k_q - k_{q-1})/J$ nor $1 - k_m/J$ is close to zero, in other words, in the region where the impurities are well separated from each other. We show in appendix B that the expression (A.4) is in fact justifiable also when two of k_q are close together. Using (A.4), we find that the large J limit of the operators in the J-summation of (A.2) is given by

$$\frac{t^{J+m} \delta_{\sum_{q=1}^m n_q, 0}}{(J+m-1)!} \sum_{0 \le k_2 \le k_3 \le \cdots \le k_m \le J} \mathrm{Tr}\left[\mathcal{O}_{M_1} Z^{k_2} \mathcal{O}_{M_2} Z^{k_3-k_2} \cdots \right.$$
$$\left. \cdots \mathcal{O}_{M_{m-1}} Z^{k_m-k_{m-1}} \mathcal{O}_{M_m} Z^{J-k_m}\right](x) e^{2\pi i \sum_{q=2}^m n_q k_q/J}. \quad \text{(A.5)}$$

Summing over the permutations of $\{\mathcal{O}_{M_2}, \mathcal{O}_{M_3}, \cdots, \mathcal{O}_{M_m}\}$, we obtain the BMN operator given in table 1 for a general m.

B. Beyond the saddle point approximation

In section 3 and appendix A, we gave analyses using the saddle point method. Thus the validity of the expressions (3.16) and (A.4) is limited only to the region where none of $(k_q - k_{q-1})/J$ nor $1 - k_m/J$ is close to zero. However, we can show that this restriction that the impurities must be well separated can in fact be relaxed; (A.4) is valid also in the case where two of the m impurities come close to each other. In this appendix we show this in the case $m = 2$. Generalization to an arbitrary m is straightforward.

Let us consider $F_2(n_2, k_2, J)$ given by (A.3) with $m = 2$ in the case of $1 - k_2/J \sim 0$ (the other case of $k_2/J \sim 0$ can be treated quite similarly). Taylor-expanding the phase factor in F_2, $e^{2\pi i n_2 \tilde{s}_2} = \sum_K (2\pi i n_2 \tilde{s}_2)^K/K!$, we have

$$F_2(n_2, k_2, J) = \sum_K \frac{1}{K!}(2\pi i n_2)^K \frac{1}{(J-k_2)!k_2!} B(k_2 + K + 1, J - k_2 + 1)\,, \tag{B.1}$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ is the beta function. Then we obtain the expansion of $F_2(n_2, k_2, J)$ in powers of $1/J$ as follows (note that $k_2/J = O(1)$ in the present case):

$$\begin{aligned}
F_2(n_2, k_2, J) &= \frac{1}{JJ!}\sum_{K=0}^{\infty}\frac{1}{K!}(2\pi i n_2)^K \left(\frac{k_2}{J}\right)^K \frac{(1 + K/k_2)\cdots(1 + 1/k_2)}{(1 + (K + 1)/J)\cdots(1 + 1/J)} \\
&= \frac{1}{JJ!}\sum_{K=0}^{\infty}\frac{1}{K!}(2\pi i n_2)^K \left(\frac{k_2}{J}\right)^K \left\{1 + \frac{1}{J}\left(\frac{J}{k_2}\frac{K(K+1)}{2} - \frac{(K+1)(K+2)}{2}\right) + \right. \\
&\qquad \left. + O(J^{-2})\right\} \\
&= \frac{1}{JJ!} e^{2\pi i n_2 k_2/J}\left\{1 + \frac{1}{J}\left(-2\pi^2 n_2^2 \frac{k_2}{J}\left(1 - \frac{k_2}{J}\right) + 2\pi i n_2\left(1 - 2\frac{k_2}{J}\right) - 1\right) + \right. \\
&\qquad \left. + O(J^{-2})\right\}.
\end{aligned} \tag{B.2}$$

The first term in the last line of (B.2) agrees with (A.4) with $m = 2$, and the next $1/J$ term is multiplied by a finite function of k_2/J. In the other case of $k_2/J \sim 0$, we obtain exactly the same expression as (B.2) by starting with the expansion $e^{2\pi i n_2 \tilde{s}_2} = \sum_K \big(-2\pi i n_2(1 - \tilde{s}_2)\big)^K/K!$. Eq. (B.2) shows that, so long as J is large, (3.17) is valid even when the two impurities are close to each other. By the same argument we can validate (A.4) for the case in which two of the m impurities come close to each other.

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JHEP06(2005)050

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JHEP06(2005)050